Abstract

The subject of this Ph.D. thesis is the mathematical Radon transform, which is well suited for curve detection in digital images, and for reconstruction of tomography images. The thesis is divided into two main parts.

Part I describes the Radon- and the Hough-transform and especially their discrete approximations with respect to curve parameter detection in digital images. The sampling relationships of the Radon transform is reviewed from a digital signal processing point of view. The discrete Radon transform is investigated for detection of curves, and aspects regarding the performance of the Radon transform assuming various types of noise is covered. Furthermore, a new fast scheme for estimating curve parameters is presented.

Part II of the thesis describes the inverse Radon transform in 2D and 3D with focus on reconstruction of tomography images. Some of the direct reconstruction schemes are analyzed, including their discrete implementation. Furthermore, several iterative reconstruction schemes based on linear algebra are reviewed and applied for reconstruction of Positron Emission Tomography (PET) images. A new and very fast implementation of 2D iterative reconstruction methods is devised. In a more practical oriented chapter, the noise in PET images is modelled from a very large number of measurements.

Several packages for Radon- and Hough-transform based curve detection and direct/iterative 2D and 3D reconstruction have been developed and provided for free.

Resume på dansk (Abstract in Danish)

Emnet for denne Ph.D. afhandling er den matematiske Radontransformation, der er velegnet til detektion af kurver i digitale billeder og til rekonstruktion af tomografiske billeder. Afhandlingen er opbygget i to dele.

Del I beskriver Radon- og Hough-transformationen og specielt deres diskrete approximationer med henblik på estimation af kurve parametre i digitale billeder. Der er beskrevet samplings-relationer for Radontransformationen ud fra et digital signalbehandlingssynspunkt. Den diskrete Radontransformation er undersøgt med henblik på detektion af kurver, og der er behandlet aspekter vedrørende metodens anvendelighed under antagelse af forskellige typer støj. Desuden er præsenteret en ny og hurtig metode for estimation af kurve parametre.

Del II af afhandlingen beskriver den inverse Radontransformation i 2D og 3D med fokus på rekonstruktion af tomografiske billeder. Flere af de direkte rekonstruktionsmetoder er analyseret inklusiv deres diskrete implementering. Desuden er der gennemgået en række lineær algebra baserede iterative rekonstruktionsmetoder, og de er anvendt til rekonstruktion af Positron Emission Tomografi (PET) billeder. En ny og meget hurtig implementering af 2D iterative rekonstruktionsmetoder er foreslået. I et mere praktisk orienteret kapitel er støj i PET billeder modelleret ud fra et stort antal målinger.

Et sæt programpakker er blevet udviklet til Radon- og Hough-transformation baseret detektion af kurve parametre og til direkte/iterativ 2D og 3D rekonstruktion, og de bliver stillet gratis til rådighed.

Preface

The Ph.D. project has been carried out from March 1, 1993 to May 31, 1996 at the Department of Mathematical Modelling (before January 1, 1996 Electronics Institute), Technical University of Denmark with supervisors John Aasted Sørensen and Peter Koefoed Møller.

The image on the front page shows the surface of a brain generated by PET scanning. The measured sinograms have been reconstructed and the brain volume has been shown using a 3D visualization package.

Contents

This Ph.D. thesis entitled *The Radon Transform - Theory and Implementation* is divided into two main parts. Part I consists of Chapters 1 to 5 and Part II of Chapters 6 to 11. Appendices are collected in Part III, and finally Part IV contains the papers submitted to journals and conferences.

In Chapter 1, the Radon transform is presented in the form used within seismics. Discrete approximations are derived, and it is shown that the Radon transform is well suited for curve parameter estimation, and in this chapter a new way of analyzing sampling relationships is introduced. Several properties are presented along with a set of examples using discrete Radon transformation. Optimization strategies for implementation of the discrete Radon transform are given, and some of the limitations concerning the allowed interval of slopes are also presented. A way to circumvent this restriction is also given.

Another way of defining the Radon transform (using normal parameters) is used in Chapter 2, and sampling relationships are derived. It is shown how this form of the Radon transform is related to the form analyzed in Chapter 1, and that the two definitions mainly cover different types of images. In Chapter 2, the images are assumed quadratic and the lines can have arbitrary orientation.

A very popular Radon-like transform is the Hough transform, which is described in Chapter 3. Possibilities, limitations, and an optimization strategy are given along with a set of examples. Here it is also shown that the discrete Hough transform is identical to the discrete Radon transform, if some of the sampling parameters are restricted.

The Radon and the Hough transforms are generalized in order to handle more general parameterized curve types. The properties of the two transforms are then exploited in the FCE-algorithm [1, 2], which is proposed for fast curve parameter estimation. The potential of the algorithm is demonstrated in two examples.

One of the very strong features of the discrete Radon transform regards noise suppression, which is covered theoretically in Chapter 5. A novel analysis of the influence of both additive noise [3] and uncertainty on the line samples [4] is presented.

In Chapter 6 the thesis changes its aim and describes computerized tomography with respects to reconstruction of PET- and CT-images. A simplified description of the fundamental physics is given and it is motivated why the inverse Radon transform can be used for reconstruction of the measured sinograms.

Several of the common direct inverse Radon schemes are derived in Chapter 7. First using normal parameter, and later in this chapter similar inversion schemes are derived for slant stacking.

The implementation of the Radon based reconstruction methods impose the use of several different elements which are reviewed from a digital signal processing point of view in Chapter 8, but only for the Radon transform using normal parameters. Chapter 8 also includes a set of examples, made with a developed software package. This and other developed software packages are provided for free.

A very different approach for developing reconstruction algorithms is based on linear algebra and statistics. In Chapter 9 the basis of these methods are shown, and the relationship with that the direct reconstruction methods is reviewed. This chapter illustrates that a broad field of research areas have contributed directly or indirectly to the field of reconstruction methods. Iterative reconstruction methods are reviewed and a very fast implementation of 2D iterative reconstruction algorithms [5, 6] is proposed. A set of examples are included, where PET-images (or PET-like images) are reconstructed from noisy sinograms and the performance of the 2D fast iterative reconstruction package is reviewed.

Next Chapter 10, goes into reconstruction of volumes using 3D PET scanners. Some of the Radon transform based reconstruction methods are derived and some of the implementation aspects are reviewed. A software package has been developed, where Radon based and iterative reconstruction methods have been implemented. It is shown that most of the methods can be implemented efficiently on a parallel computer, and a few examples are presented.

The final chapter in the main thesis is Chapter 11, which is of a more practical nature. From a huge set of measurements on phantoms and humans the noise in reconstructed PET images has been modelled and model parameters have been estimated [7, 8].

It should be mentioned that a part of the work done in this project is far better presented using the World Wide Web tools of movies and virtual reality objects. It has been chosen to avoid color images in the thesis, even though that colors normally will enhance the visual impression. MPEG movies and 3D virtual objects can be found at the Human Brain Project WWW-server [9]. This thesis is available as a Postscript file, which can be down-loaded from [10].

Collaborations

Chapter 4 present work made in collaboration with Kim Vejlby Hansen. The work has been carried out as a joint venture project between Ødegaard & Danneskiold-Samsøe and Department of Mathematical Modelling (before January 1, 1996 Electronics Institute), Technical University of Denmark. The ideas and results have been presented at the *EUSIPCO Conference 94* in Edinburgh, Scotland, and at the *Interdisciplinary Inversion Summer School 94* in Mønsted, Denmark, and published in [1, 2]. These papers are shown in Appendices G and H. Kim Vejlby Hansen and I had a very long and good collaboration. He and Peter Koefoed Møller are thanked for getting me into the area of the Radon transform in the first place.

Chapters 6, 7, and 8 are inspired by my stay at the National University Hospital in Copenhagen (Rigshospitalet) and by two masters projects carried out at that time. Software packages have been made for 2D direct reconstruction of PET images, and one for analytical generation of sinograms from a set of primitives. These packages can be used for quantifying the quality of reconstruction algorithms. I will like to thank Claus Svarer and Karin Stahr for making our stay

very good, and especially Søren Holm with whom I have had a long and fruitful collaboration. I have learned much about PET and tomography in general from him. My former students Peter Philipsen and Jesper James Jensen are acknowledged for their collaboration and hard work. We have spent many joyful hours together, and their efforts have meant much to me.

In Chapter 9 the fundamentals of linear algebra based reconstruction methods are covered with special focus on the implementation of iterative reconstruction methods. For this work I had the pleasure of working with Jesper James Jensen. We developed a very fast technique for implementing most 2D iterative reconstruction methods. This work has been submitted in [5] and [6], shown in Appendix L and M, respectively.

Some of the functions in the 3D reconstruction package presented in Chapter 10 has been made by Peter Philipsen and he helped by adding the compiler options needed to speed up the program on the Onyx-computer from Silicon Graphics (SGI).

Chapter 11 covers recent work made together with Søren Holm, where noise in PET reconstructed images has been modelled and the model parameters have been estimated from a huge set of measurements. The results have been presented at the *IEEE Medical Imaging Conference 95* in San Francisco USA, and published in a short version [7], shown in Appendix J, and submitted in a longer version in [8], shown in Appendix K.

In Appendix N the published papers [11, 12] are shown, where mean field techniques have been used to improve image quality by using strong priors in the restoration of PET reconstructed images. This work was made with Lars Kai Hansen and Peter Philipsen. It has been presented at the *Fourth Danish Conference for Pattern Recognition and Image Analysis 95* in Copenhagen, Denmark and at the *Interdisciplinary Inversion Conference 95* in Århus, Denmark.

Acknowledgments

I thank my two supervisors, Peter Koefoed Møller and John Aasted Sørensen for getting me into the project, their support during the years, and giving me very free limits, which I have enjoyed.

I am very grateful to Lars Kai Hansen for an inspiring collaboration and for his commitment to create a pleasant and dynamical environment at the department. He got me interested in medical imaging and laid many bricks along the way.

My room mates Søren Kragh Jespersen, Cyril Goutte, and Peter S. K. Hansen are acknowledged for their proofreading and support.

My wife Katja has directly and indirectly contributed to my work, and without her loving support I could never have managed getting this work done.

Thanks to the many WWW users, who have responded on my home page. Also thank you to all the programmers contributing to the Linux project.

Finally thanks to the other current and earlier Ph.D. students and staff at the Electronics Institute and Department of Mathematical Modelling, especially Jan Larsen, with whom I have had many hard and good discussions, Torsten Lehmann for introducing and helping me to use Linux in the early days, Mogens Dyrdal for support, and Simon Boel Pedersen for his friendly attitude and fantastic lectures in Digital Signal Processing.

Papers

During the project a total of 13 reports have been made, mainly for teaching purposes, and additionally several parts of this thesis has been submitted to conferences or journals:

- **P. A. Toft** and K. V. Hansen: *"Fast Radon Transform for Detection of Seismic Reflections"*, Signal Processing VII - Theories and Applications. Proceedings of EUSIPCO 94, pages 229-232. Shown in Appendix G.
- K. V. Hansen and **P. A. Toft**: *"Fast Curve Estimation Using Pre-Conditioned Generalized Radon Transform"*. Accepted for publication in IEEE Transactions on Image Processing. Shown in Appendix H.
- **Peter Toft**: *"Using the Generalized Radon Transform for Detection of Curves in Noisy Images"*. Proceedings of the IEEE ICASSP 1996, pages 2221-2225 in part IV. Shown in Appendix I.
- Søren Holm, **Peter Toft**, and Mikael Jensen: *"Estimation of the noise contributions from Blank, Transmission and Emission scans in PET"*. Accepted for publication in the Conference Issue of IEEE Medical Imaging Conference 95. Furthermore, the corresponding abstract can be found in the abstract collection of Medical Imaging Conference 95. Shown in Appendix J.
- Søren Holm, **Peter Toft**, and Mikael Jensen: *"Estimation of the noise contributions from Blank, Transmission and Emission scans in PET"*. Submitted to IEEE Transactions on Nuclear Science. Shown in Appendix K.
- **Peter Toft** and Jesper James Jensen: *"A very fast implementation of 2D Iterative Reconstruction Algorithms"*. Submitted to IEEE Medical Imaging Conference 1996. Summary and abstract can be found in Appendix L.
- **Peter Toft** and Jesper James Jensen: *"Accelerated 2D Iterative Reconstruction"*. Submitted to IEEE Transactions on Medical Imaging. Shown in Appendix M.
- Peter Alshede Philipsen, Lars Kai Hansen and **Peter Toft**: *"Mean Field Reconstruction with Snaky Edge Hints"*. Accepted for publication in the book *"INVERSE METHODS - Interdisciplinary elements of Methodology, Computation and Application"*. Will be published by Springer-Verlag in 1996. An (almost) identical paper can be found in Proceedings of the *Fourth Danish Conference on Pattern Recognition and Image Analysis 95*, pages 155-161. This paper can be found in Appendix N.
- **Peter Toft**: *"Detection of Lines with Wiggles using the Radon Transform"*. Submitted to to the *NORSIG'96 - 1996 IEEE Nordic Signal Processing Symposium* in Espoo, Finland. This paper can be found in Appendix O.

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Style Conventions

- References to the bibliography placed in the end of the thesis will appear as [13] or [14, 15].
- Equation and Figure has been abbreviated to Eq. and Fig. Likewise Equations and Figures to Eqs. and Figs.
- Equations, tables, and figures are numbered by the chapter, i.e., Eq. 10.2 is the second equation in Chapter 10.
- The appendices are numbered using capitals, starting with Appendix A.
- Normally the letters m, n, k, h, l are used to denote integer type variables, and the letters x, y, z denote continuous variables.
- Vectors are denoted by bold-faced small letters or Greek letters, like $\boldsymbol{b}$ or $\boldsymbol{\xi}$.
- Transpose of a vector $\boldsymbol{b}$ is shown as $\boldsymbol{b}^T$.
- Matrices are denoted by bold-faced capital letters, like $\boldsymbol{A}$. A real valued matrix with I rows and J columns are denoted $\boldsymbol{A} \in \mathbb{R}^{I \times J}$ The individual elements are $a_{i,j}$ corresponding to row i and column j.
- Vectors are always column vectors, and the i'th rows of the matrix are denoted $\boldsymbol{a}_i$, i.e.,

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1^T \\ \boldsymbol{a}_2^T \\ \dots \\ \boldsymbol{a}_i^T \\ \dots \\ \boldsymbol{a}_I^T \end{bmatrix} \tag{0.1}$$

- Program examples in pseudo C-code will use the sans serif font and symbols using that font refers to the pseudo-code. An examples of pseudo-code are called an Algorithm, which looks like

ALGORITHM 0.1 : SMALL EXAMPLE

```
For k = 0 to K-1                          //Help is placed like this
  p(k) = p_min+k*Delta_p                  //Multiplication in algorithms use *
End
```

- Vector indices in equations starts from 1, but in order to aid implementation i C or C++ the indices start from 0 in the pseudo code. Given that the pseudo code is intended for overview, this should not lead to confusion.
- The delta function is denoted by $\delta(\cdot)$. This function is reviewed in Appendix A.
- The Hamilton step function is denoted by $\mu(\cdot)$. The function is zero if the argument is negative, and one if the argument is positive.
- Rounding to the nearest upper integer, $\lceil\cdot\rceil$.
- Rounding to the nearest lower integer, $\lfloor\cdot\rfloor$. In an algorithm the function floor is used.
- Rounding to the nearest integer, $[\cdot]$. In an algorithm the function round is used.
- Fourier transform pairs are marked as $g(t) \leftrightarrow G(f)$.
- In equation convolutions are denoted by $*$ for a one dimensional, $**$ for a two-dimensional, and $***$ for a three-dimenisonal convolution.
- In the algorithms, i.e., pseudo-code, no convolutions are found and the symbol $*$ will be used as a simple multiplication.
- The scalar product between to (column) vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are denoted by $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^T \boldsymbol{b}$.
- The symbol $\forall$ denotes 'for all'.
- The length of vectors are normally denoted by $|\cdot|$, but in Chapter 9 the linear algebra notation $\|\cdot\|_2$ has been used, given by $\|\boldsymbol{v}\|_2 = \sqrt{\boldsymbol{v}^t\boldsymbol{v}} = \sqrt{\sum_j v_j^2}$.

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